Review Problems for the Final

These problems are provided to help you study. The presence of a problem on this handout does not imply that there will be a similar problem on the test. And the absence of a topic does not imply that it won’t appear on the test.

1. Find the area of the intersection of the interiors of the circles

\[ x^2 + (y - 1)^2 = 1 \quad \text{and} \quad (x - \sqrt{3})^2 + y^2 = 3. \]

2. Does the series

\[ \frac{1}{2} - \frac{4}{2^3 + 1} + \frac{9}{3^3 + 1} - \frac{16}{4^3 + 1} + \cdots \]

converge absolutely, converge conditionally, or diverge?

3. Find the sum of the series

\[ \frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots. \]

4. In each case, determine whether the series converges or diverges.

(a) \[ \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{4/3}}. \]

(b) \[ \frac{2}{1} + \frac{2 \cdot 5}{1 \cdot 5} + \cdots + \frac{2 \cdot 5 \cdots (3n - 1)}{1 \cdot 5 \cdots (4n - 3)}. \]

(c) \[ \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}. \]

(d) \[ \frac{2}{3} - \frac{5}{8} + \frac{8}{13} - \frac{11}{18} + \cdots. \]

(e) \[ \sum_{n=1}^{\infty} \frac{3n^2 + 4n + 2}{\sqrt{n^3 + 16}}. \]

(f) \[ \sum_{n=1}^{\infty} \frac{5 + \cos(e^n)}{n}. \]

5. Find the values of \( x \) for which the series

\[ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} (x - 5)^n \]

converges absolutely.

6. Compute the following integrals.

(a) \[ \int e^x \cos 2x \, dx. \]

(b) \[ \int \frac{x^2}{\sqrt{4 - x^2}} \, dx. \]
(c) \[ \int \frac{5x^2 - 6x - 5}{(x - 1)^2(x + 2)} \, dx. \]

(d) \[ \int (\sin 4x)^3 (\cos 4x)^2 \, dx. \]

(e) \[ \int (\sin 4x)^2 (\cos 4x)^2 \, dx. \]

(f) \[ \int \frac{1}{(-3 - 4x - x^2)^{3/2}} \, dx. \]

7. Let \( R \) be the region bounded above by \( y = x + 2 \), bounded below by \( y = -x^2 \), and bounded on the sides by \( x = -2 \) and by the y-axis. Find the volume of the solid generated by revolving \( R \) about the line \( x = 1 \).

8. Compute \[ \lim_{x \to \infty} \left( \sqrt{x^2 + 8x - x} \right). \]

9. Compute \[ \lim_{x \to \infty} \left( 1 + \frac{2}{x} \right)^{3x}. \]

10. If \( x = t + e^t \) and \( y = t + t^3 \), find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) at \( t = 1 \).

11. (a) Find the Taylor expansion at \( c = 1 \) for \( e^{2x} \).

(b) Find the Taylor expansion at \( c = 1 \) for \( \frac{1}{3 + x} \). What is the interval of convergence?

12. Find the area of the region which lies between the graphs of \( y = x^2 \) and \( y = x + 2 \), from \( x = 1 \) to \( x = 3 \).

13. Find the area of the region between \( y = x + 3 \) and \( y = 7 - x \) from \( x = 0 \) to \( x = 3 \).

14. The base of a solid is the region in the \( x-y \)-plane bounded above by the curve \( y = e^x \), below by the \( x \)-axis, and on the sides by the lines \( x = 0 \) and \( x = 1 \). The cross-sections in planes perpendicular to the \( x \)-axis are squares with one side in the \( x-y \)-plane. Find the volume of the solid.

15. Find the interval of convergence of the power series \[ \sum_{n=1}^{\infty} \frac{(x - 3)^n}{n(2^n)^3}. \]

16. Find the slope of the tangent line to the polar curve \( r = \sin 2\theta \) at \( \theta = \frac{\pi}{6} \).

17. A tank built in the shape of the bottom half of a sphere of radius 2 feet is filled with water. Find the work done in pumping all the water out of the top of the tank.

18. Let \[ x = \frac{\sqrt{3}}{2} t^2, \quad y = t - \frac{1}{4} t^3. \]

Find the length of the arc of the curve from \( t = -2 \) to \( t = 2 \).

19. Find the area of the surface generated by revolving \( y = \frac{1}{3} x^3 \), \( 0 \leq x \leq 2 \), about the \( x \)-axis.

20. (a) Convert \((x - 3)^2 + (y + 4)^2 = 25\) to polar and simplify.

(b) Convert \( r = 4 \cos \theta - 6 \sin \theta \) to rectangular and describe the graph.

21. Find the area of the region inside the cardioid \( r = 1 + \cos \theta \) and outside the circle \( r = 3 \cos \theta \).
Solutions to the Review Problems for the Final

1. Find the area of the intersection of the interiors of the circles

\[ x^2 + (y - 1)^2 = 1 \quad \text{and} \quad (x - \sqrt{3})^2 + y^2 = 3. \]

Convert the two equations to polar:

\[ x^2 + y^2 - 2y + 1 = 1, \quad x^2 + y^2 = 2y, \quad r^2 = 2r \sin \theta, \quad r = 2 \sin \theta. \]

\[ (x - \sqrt{3})^2 + y^2 = 3, \quad x^2 - 2\sqrt{3}x + 3 + y^2 = 3, \quad x^2 + y^2 = 2\sqrt{3}x, \quad r^2 = 2\sqrt{3}r \cos \theta, \quad r = 2\sqrt{3} \cos \theta. \]

Set the equations equal to solve for the line of intersection:

\[ 2 \sin \theta = 2\sqrt{3} \cos \theta, \quad \tan \theta = \sqrt{3}, \quad \theta = \frac{\pi}{3}. \]

The region is “orange-slice”-shaped, with the bottom/right half bounded by \( r = 2 \sin \theta \) from \( \theta = 0 \) to \( \theta = \frac{\pi}{3} \) and the top/left half bounded by \( r = 2\sqrt{3} \cos \theta \) from \( \theta = \frac{\pi}{3} \) to \( \theta = \frac{\pi}{2} \). Hence, the area is

\[
A = \int_{0}^{\pi/3} \frac{1}{2} (2 \sin \theta)^2 d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} (2\sqrt{3} \cos \theta)^2 d\theta = 2 \int_{0}^{\pi/3} (\sin \theta)^2 d\theta + 6 \int_{\pi/3}^{\pi/2} (\cos \theta)^2 d\theta = 
\]

\[
\int_{0}^{\pi/3} (1 - \cos 2\theta) d\theta + 3 \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta = \left[ \theta + \frac{1}{2} \sin 2\theta \right]_{0}^{\pi/3} + 3 \left[ \theta + \frac{1}{2} \sin 2\theta \right]_{\pi/3}^{\pi/2} = 
\]

\[
\frac{5}{6} \pi - 3 \approx 0.88594. \quad \Box
\]

2. Does the series

\[ \frac{1}{2} - \frac{4}{2^3 + 1} + \frac{9}{3^3 + 1} - \frac{16}{4^3 + 1} + \cdots \]

converge absolutely, converge conditionally, or diverge?

\[
\frac{1}{2} - \frac{4}{2^3 + 1} + \frac{9}{3^3 + 1} - \frac{16}{4^3 + 1} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1}.
\]
The absolute value series is
\[ \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}. \]

Since
\[ \frac{n^2}{n^3 + 1} \approx \frac{1}{n} \]
for large values of \( n \), I’ll compare the series to \( \sum_{n=1}^{\infty} \frac{1}{n} \).

\[
\lim_{n \to \infty} \frac{n^2}{n^3 + 1} = \lim_{n \to \infty} \frac{n^3 + 1}{n} = 1.
\]

The limit is finite \((\neq \infty)\) and positive \((>0)\). The harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. By Limit Comparison, the series \( \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1} \) diverges. Hence, the original series does not converge absolutely.

Returning to the original series, note that it alternates, and
\[
\lim_{n \to \infty} \frac{n^2}{n^3 + 1} = 0.
\]

Let \( f(n) = \frac{n^2}{n^3 + 1} \). Then
\[
f'(n) = \frac{n(2 - n^3)}{(1 + n^3)^2} < 0
\]
for \( n > 1 \). Therefore, the terms of the series decrease for \( n \geq 2 \), and I can apply the Alternating Series Rule to conclude that the series converges. Since it doesn’t converge absolutely, but it \textit{does} converge, it converges conditionally.

3. Find the sum of the series
\[
\frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots.
\]
\[
= \frac{5}{9} \left( 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots \right) = \frac{5}{9} \cdot \frac{1}{1 - \left( -\frac{1}{3} \right)} = \frac{5}{9} \cdot \frac{3}{4} = \frac{5}{12}.
\]

4. In each case, determine whether the series converges or diverges.

(a) \( \sum_{n=2}^{\infty} \frac{1}{n (\ln n)^{4/3}} \).

Apply the Integral Test. The function \( f(n) = \frac{1}{n (\ln n)^{4/3}} \) is positive and continuous on the interval \([2, +\infty)\).

Since
\[
f'(n) = -\frac{4}{3n^2 (\ln n)^{7/3}} - \frac{1}{n^2 (\ln n)^{4/3}},
\]
it follows that \( f'(n) < 0 \) for \( n \geq 2 \). Hence, \( f \) decreases on the interval \([2, +\infty)\). The hypotheses of the Integral Test are satisfied.
Compute the integral:
\[
\int_{2}^{\infty} \frac{1}{n(\ln n)^{4/3}} \, dn = \lim_{p \to \infty} \int_{2}^{p} \frac{1}{n(\ln n)^{4/3}} \, dn = \lim_{p \to \infty} \left[ -3 \frac{1}{(\ln n)^{1/3}} \right]_{2}^{p} = -3 \lim_{p \to \infty} \left( \frac{1}{(\ln p)^{1/3}} - \frac{1}{(\ln 2)^{1/3}} \right) = \frac{3}{(\ln 2)^{1/3}}.
\]

(To do the integral, I substituted \( u = \ln n \), so \( du = \frac{1}{n} \, dn \).)

Since the integral converges, the series converges, by the Integral Test.

(b) \[ \frac{1}{1} + \frac{2}{1 \cdot 5} + \cdots + \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 5 \cdots (4n-3)}. \]

Apply the Ratio Test. The \( n \)-th term of the series is
\[ a_n = \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 5 \cdots (4n-3)}, \]
so the \((n+1)\)-st term is
\[ a_{n+1} = \frac{2 \cdot 5 \cdots (3n-1) \cdot (3n+1-1)}{1 \cdot 5 \cdots (4n-3) \cdot (4n+1-3)}. \]

Hence,
\[
a_{n+1} = \frac{2 \cdot 5 \cdots (3n-1) \cdot (3n+1-1) \cdot 1 \cdot 5 \cdots (4n-3) \cdot (4n+1) \cdot (3n+1-1)}{2 \cdot 5 \cdots (3n-1) \cdot 1 \cdot 5 \cdots (4n-3) \cdot (4n+1) \cdot (4n+1-3)} = \frac{3(n+1)-1}{4(n+1)-3} = \frac{3n+2}{4n+1}.
\]
The limiting ratio is
\[
\lim_{n \to \infty} \frac{3n+2}{4n+1} = \frac{3}{4}.
\]
The limit is less than 1, so the series converges, by the Ratio Test.

(c) \[ \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^{-n^2}. \]

Apply the Root Test.
\[ a_n^{1/n} = \left( 1 + \frac{1}{n} \right)^{-n}. \]
The limit is
\[
\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{-n} = \lim_{n \to \infty} \left( \left( 1 + \frac{1}{n} \right)^n \right)^{-1} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{-n} = e^{-1}.
\]
Since \( e^{-1} = \frac{1}{e} < 1 \), the series converges, by the Root Test.

(d) \[ \frac{2}{3} - \frac{5}{8} + \frac{8}{13} - \frac{11}{18} + \cdots. \]

Since
\[ \lim_{n \to \infty} \frac{2 + 3n}{3 + 5n} = \frac{3}{5}, \]
it follows that \( \lim_{n \to \infty} a_n \) is undefined — the values oscillate, approaching \( \pm \frac{3}{5} \). Since, in particular, the limit is nonzero, the series diverges, by the Zero Limit Test.
(e) \[ \sum_{n=1}^{\infty} \frac{3n^2 + 4n + 2}{\sqrt{n^5 + 16}}. \]

Apply Limit Comparison:

\[
\lim_{n \to \infty} \frac{3n^2 + 4n + 2}{\sqrt{n^5 + 16}} = \lim_{n \to \infty} \frac{3n^{5/2} + 4n^{3/2} + 2n^{1/2}}{\sqrt{n^5 + 16}} = 3.
\]

The limit is finite and positive. The series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges, because it’s a p-series with \( p = \frac{1}{2} < 1 \). Therefore, the original series diverges by Limit Comparison.

(f) \[ \sum_{n=1}^{\infty} \frac{5 + \cos(e^n)}{n}. \]

\[-1 \leq \cos(e^n) \leq 1 \quad \frac{4}{n} \leq \frac{5 + \cos(e^n)}{n} \leq \frac{6}{n} \]

\[ \sum_{n=1}^{\infty} \frac{4}{n} \] diverges, because it’s 4 times the harmonic series. Therefore, \[ \sum_{n=1}^{\infty} \frac{5 + \cos(e^n)}{n} \] diverges by Direct Comparison.

5. Find the values of \( x \) for which the series

\[ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} (x - 5)^n \]

converges absolutely.

Apply the Ratio Test:

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{((n+1)!)^2 |x - 5|^{n+1}}{(2(n+1))!} \cdot \frac{(n!)^2 |x - 5|^n}{(2n)! |x - 5|} = \left( \frac{(n+1)!}{n!} \right)^2 \left( \frac{(2n+1)!}{(2n+2)!} \right) |x - 5| = \frac{(n+1)^2}{(2n+1)(2n+2)} |x - 5|.
\]

The limiting ratio is

\[
\lim_{n \to \infty} \frac{(n+1)^2}{(2n+1)(2n+2)} |x - 5| = \frac{1}{4} |x - 5|.
\]

The series converges absolutely for \( \frac{1}{4} |x - 5| < 1 \), i.e. for \( 1 < x < 9 \). The series diverges for \( x < 1 \) and for \( x > 9 \).

You’ll probably find it difficult to determine what is happening at the endpoints! However, if you experiment — compute some terms of the series for \( x = 9 \), for instance — you’ll see that the individual terms are growing larger, so the series at \( x = 1 \) and at \( x = 9 \) diverge, by the Zero Limit Test.

6. Compute the following integrals.
(a) \( \int e^x \cos 2x \, dx \).

\[
\frac{d}{dx} \int dx + e^x \cos 2x - e^x \sin 2x + e^x \rightarrow -\frac{1}{4} \cos 2x
\]

\[
\int e^x \cos 2x \, dx = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4} \int e^x \cos 2x \, dx,
\]

\[
\frac{5}{4} \int e^x \cos 2x \, dx = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x,
\]

\[
\int e^x \cos 2x \, dx = \frac{2}{5} e^x \sin 2x + \frac{1}{5} e^x \cos 2x + C. \quad \Box
\]

(b) \( \int \frac{x^2}{\sqrt{4-x^2}} \, dx \).

\[
\int \frac{x^2}{\sqrt{4-x^2}} \, dx = \int \frac{4 \sin^2 \theta}{\sqrt{4-4 \sin^2 \theta}} \, 2 \cos \theta \, d\theta = \int \frac{4 \sin^2 \theta}{\sqrt{4 \cos^2 \theta}} \, 2 \cos \theta \, d\theta = 4 \int \sin^2 \theta \, d\theta =
\]

\[
[ x = 2 \sin \theta, \quad dx = 2 \cos \theta \, d\theta ]
\]

\[2 \int (1 - \cos 2 \theta) \, d\theta = 2 \left( \theta - \frac{1}{2} \sin 2 \theta \right) + C = 2 (\theta - \sin \theta \cos \theta) + C = 2 \sin^{-1} \frac{x}{2} - \frac{1}{2} x \sqrt{4-x^2} + C.
\]

(c) \( \int \frac{5x^2 - 6x - 5}{(x-1)^2(x+2)} \, dx \).

\[
\frac{5x^2 - 6x - 5}{(x-1)^2(x+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2},
\]

\[
5x^2 - 6x - 5 = a(x-1)(x+2) + b(x+2) + c(x-1)^2.
\]

Setting \( x = 1 \) gives \(-6 = 3b, \) so \( b = -2.\)

Setting \( x = -2 \) gives \( 27 = 9c, \) so \( c = 3.\)

Therefore, \( 5x^2 - 6x - 5 = a(x-1)(x+2) - 2(x+2) + 3(x-1)^2.\)

Setting \( x = 0 \) gives \(-5 = -2a - 4 + 3, \) so \( a = 2.\)

Thus,

\[
\int \frac{5x^2 - 6x - 5}{(x-1)^2(x+2)} \, dx = \int \left( \frac{2}{x-1} - \frac{2}{(x-1)^2} + \frac{3}{x+2} \right) \, dx = 2 \ln |x-1| + \frac{2}{x-1} + 3 \ln |x+2| + C. \quad \Box
\]
(d) \[ \int (\sin 4x)^3 (\cos 4)^2 \, dx. \]

\[
\int (\sin 4x)^3 (\cos 4)^2 \, dx = \int (\sin 4x)^2 (\cos 4)^2 (\sin 4 \, dx) = \int (1 - (\cos 4x)^2)^2 (\cos 4x) \, dx = \\
\left[ u = \cos 4x, \quad du = -4 \sin 4x \, dx, \quad dx = \frac{du}{-4 \sin 4x} \right] \\
\int (1 - u^2) u^2 (\sin 4x) \left( \frac{du}{-4 \sin 4x} \right) = \frac{1}{4} \int (u^4 - u^2) \, du = \frac{1}{4} \left( \frac{1}{5}u^5 - \frac{1}{3}u^3 \right) + C = \\
\frac{1}{4} \left( \frac{1}{5}(\cos 4x)^5 - \frac{1}{3}(\cos 4x)^3 \right) + C. \quad \Box
\]

(e) \[ \int (\sin 4x)^3 (\cos 4)^2 \, dx. \]

\[
\int (\sin 4x)^3 (\cos 4)^2 \, dx = \int \frac{1}{2} (1 - \cos 8x) \cdot \frac{1}{2} (1 + \cos 8x) \, dx = \frac{1}{4} \int (1 - (\cos 8x)^2) \, dx = \frac{1}{4} \int (\sin 8x)^2 \, dx = \\
\frac{1}{8} \int (1 - \cos 16x) \, dx = \frac{1}{8} \left( x - \frac{1}{16} \sin 16x \right) + C. \quad \Box
\]

(f) \[ \int \frac{1}{(-3 - 4x - x^2)^{3/2}} \, dx. \]

I need to complete the square. Note that \(-\frac{4}{2} = -2\) and \((-2)^2 = 4\). Then

\[-3 - 4x - x^2 = -(x^2 + 4x + 3) = -(x^2 + 4x + 4 - 1) = -(x + 2)^2 - 1 = 1 - (x + 2)^2.\]

So

\[
\int \frac{1}{(-3 - 4x - x^2)^{3/2}} \, dx = \int \frac{1}{(1 - (x + 2)^2)^{3/2}} \, dx = \int \frac{1}{(1 - (\sin \theta)^2)^{3/2}} (\cos \theta \, d\theta) = \int \frac{1}{(\cos \theta)^3} (\cos \theta \, d\theta) = \\
[x + 2 = \sin \theta, \quad dx = \cos \theta \, d\theta]
\]

\[
\int \frac{1}{(\cos \theta)^2} \, d\theta = \int (\sec \theta)^2 \, d\theta = \tan \theta + C = \frac{x + 2}{\sqrt{3 - 4x - x^2}} + C. \quad \Box
\]

7. Let \(R\) be the region bounded above by \(y = x + 2\), bounded below by \(y = -x^2\), and bounded on the sides by \(x = -2\) and by the \(y\)-axis. Find the volume of the solid generated by revolving \(R\) about the line \(x = 1\).
Most of the things in the picture are easy to understand — but why is \( r = 1 - x? \)

Notice that the distance from the y-axis to the side of the shell is \(-x\), not \(x\). Reason: \(x\)-values to the left of the y-axis are negative, but distances are always positive. Thus, I must use \(-x\) to get a positive value for the distance.

As usual, \(r\) is the distance from the axis of revolution \(x = 1\) to the side of the shell, which is \(1 + (-x) = 1 - x\).

The left-hand cross-section extends from \(x = -2\) to \(x = 0\). You can check that if you plug \(x\)'s between \(-2\) and 0 into \(r = 1 - x\), you get the correct distance from the side of the shell to the axis \(x = 1\).

The volume is

\[
V = \int_{-2}^{0} 2\pi(1-x)((x+2) + x^2) \, dx = 4\pi = \int_{-2}^{0} 2\pi \left(2 - x - x^3\right) \, dx = 2\pi \left[2x - \frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{-2}^{0} = 20\pi \approx 62.83185.
\]

8. Compute \( \lim_{x \to \infty} \left(\sqrt{x^2 + 8x - x}\right) \).

\[
\lim_{x \to \infty} \left(\sqrt{x^2 + 8x - x}\right) = \lim_{x \to \infty} \left(\sqrt{x^2 + 8x + x}\right) = \lim_{x \to \infty} \frac{x^2 + 8x - x^2}{\sqrt{x^2 + 8x + x}} = \lim_{x \to \infty} \frac{8x}{\sqrt{1 + \frac{8}{x} + 1}} = \frac{8}{1+1} = 4. \text{ \(\Box\)}
\]

9. Compute \( \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x} \).

Let \( y = \left(1 + \frac{2}{x}\right)^{3x} \), so

\[
\ln y = \ln \left(1 + \frac{2}{x}\right)^{3x} = 3x \ln \left(1 + \frac{2}{x}\right).
\]

Then

\[
\lim_{x \to \infty} \ln y = \lim_{x \to \infty} 3x \ln \left(1 + \frac{2}{x}\right) = \lim_{x \to \infty} \frac{\ln \left(1 + \frac{2}{x}\right)}{\frac{1}{3x}} = \lim_{x \to \infty} \frac{\left(\frac{1}{1 + \frac{2}{x}}\right) \left(-\frac{2}{x^2}\right)}{-\frac{1}{3x^2}} = 6 \lim_{x \to \infty} \frac{1}{1 + \frac{2}{x}} = 6.
\]

Therefore,

\[
\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x} = \lim_{x \to \infty} y = e^{\lim_{x \to \infty} \ln y} = e^6. \text{ \(\Box\)}
\]

10. If \( x = t + e^t \) and \( y = t + t^3 \), find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) at \( t = 1 \).

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + 3t^2}{1 + e^t}.
\]

9
When \( t = 1 \), \( \frac{dy}{dx} = \frac{4}{1 + e} \).

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \left( \frac{dt}{dx} \right) \left( \frac{d}{dt} \left( \frac{dy}{dx} \right) \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{1 + 3t^2}{1 + e^t} \right).
\]

When \( t = 1 \), \( \frac{d^2y}{dx^2} = \frac{6 + 2e}{(1 + e)^3} \).

11. (a) Find the Taylor expansion at \( c = 1 \) for \( e^{2x} \).

\[
e^{2x} = e^{2(x-1)+2} = e^2e^{2(x-1)} = e^2 \left( 1 + 2(x-1) + \frac{2^2(x-1)^2}{2!} + \frac{2^3(x-1)^3}{3!} + \cdots \right).
\]

(b) Find the Taylor expansion at \( c = 1 \) for \( \frac{1}{3 + x} \). What is the interval of convergence?

\[
\frac{1}{3 + x} = \frac{1}{4 + (x - 1)} = \frac{1}{4} \cdot \frac{1}{1 + \frac{x - 1}{4}} = \frac{1}{4} \cdot \frac{1}{1 - \left( -\frac{x - 1}{4} \right)} = \frac{1}{4} \left( 1 - \frac{x - 1}{4} + \left( \frac{x - 1}{4} \right)^2 - \left( \frac{x - 1}{4} \right)^3 + \cdots \right).
\]

The series converges for \(-1 < \frac{x - 1}{4} < 1\), i.e. for \(-3 < x < 5\).

12. Find the area of the region which lies between the graphs of \( y = x^2 \) and \( y = x + 2 \), from \( x = 1 \) to \( x = 3 \).

As the picture shows, the curves intersect. Find the intersection point:

\[
x^2 = x + 2, \quad x^2 - x - 2 = 0, \quad (x - 2)(x + 1) = 0, \quad x = 2 \quad \text{or} \quad x = -1.
\]
On the interval $1 \leq x \leq 3$, the curves cross at $x = 2$. I’ll use vertical rectangles. From $x = 1$ to $x = 2$, the top curve is $y = x + 2$ and the bottom curve is $y = x^2$. From $x = 2$ to $x = 3$, the top curve is $y = x^2$ and the bottom curve is $y = x + 2$. The area is

$$A = \int_1^2 ((x + 2) - x^2) \, dx + \int_2^3 (x^2 - (x + 2)) \, dx = 3.$$  

13. Find the area of the region between $y = x + 3$ and $y = 7 - x$ from $x = 0$ to $x = 3$.

As the picture shows, the curves intersect. Find the intersection point:

$$x + 3 = 7 - x, \quad 2x = 4, \quad x = 2.$$

I’ll use vertical rectangles. From $x = 0$ to $x = 2$, the top curve is $y = 7 - x$ and the bottom curve is $y = x + 3$. From $x = 2$ to $x = 3$, the top curve is $y = x + 3$ and the bottom curve is $y = 7 - x$. The area is

$$\int_0^2 ((7 - x) - (x + 3)) \, dx + \int_2^3 ((x + 3) - (7 - x)) \, dx = \int_0^2 (4 - 2x) \, dx + \int_2^3 (2x - 4) \, dx = [4x - x^2]_0^2 + [x^2 - 4x]_2^3 = 4 + 1 = 5.$$  

14. The base of a solid is the region in the $x$-$y$-plane bounded above by the curve $y = e^x$, below by the $x$-axis, and on the sides by the lines $x = 0$ and $x = 1$. The cross-sections in planes perpendicular to the $x$-axis are squares with one side in the $x$-$y$-plane. Find the volume of the solid.

The volume is

$$V = \int_0^1 (e^x)^2 \, dx = \int_0^1 e^{2x} \, dx = \left[ \frac{1}{2} e^{2x} \right]_0^1 = \frac{1}{2} (e^2 - 1) \approx 3.19453.$$
15. Find the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(x-3)^n}{n(2^n)^3} \).

Apply the Ratio Test to the absolute value series:

\[
\lim_{n \to \infty} \frac{|x-3|^{n+1}}{(n+1)(2^{n+1})^3} = \frac{n}{n+1} \left( \frac{2^n}{2^{n+1}} \right)^3 \frac{|x-3|^{n+1}}{|x-3|^n} = \frac{n}{n+1} \cdot 8|x-3| = \frac{1}{8}|x-3|.
\]

The series converges for \( \frac{1}{8}|x-3| < 1 \), i.e. for \(-5 < x < 11\).
At \( x = 11 \), the series is

\[
\sum_{n=1}^{\infty} \frac{8^n}{n(2^n)^3} = \sum_{n=1}^{\infty} \frac{1}{n}.
\]

It’s harmonic, so it diverges.
At \( x = -5 \), the series is

\[
\sum_{n=1}^{\infty} \frac{(-8)^n}{n(2^n)^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.
\]

This is the alternating harmonic series, so it converges.
Therefore, the power series converges for \(-5 \leq x < 11\), and diverges elsewhere. \( \Box \)

16. Find the slope of the tangent line to the polar curve \( r = \sin 2\theta \) at \( \theta = \frac{\pi}{6} \).

When \( \theta = \frac{\pi}{6} \), \( r = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \). Since \( \frac{dr}{d\theta} = 2\cos 2\theta \), when \( \theta = \frac{\pi}{6} \), \( \frac{dr}{d\theta} = 2 \cos \frac{\pi}{3} = 1 \).

The slope of the tangent line is

\[
\frac{dy}{dx} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}} = \frac{\left( \frac{\sqrt{3}}{2} \right) \left( \frac{\sqrt{3}}{2} \right) + \left( \frac{1}{2} \right)}{-\left( \frac{\sqrt{3}}{2} \right) \left( \frac{1}{2} \right) + \left( \frac{\sqrt{3}}{2} \right)} \approx \frac{5\sqrt{3}}{3} \approx 2.88675. \quad \Box
\]

17. A tank built in the shape of the bottom half of a sphere of radius 2 feet is filled with water. Find the work done in pumping all the water out of the top of the tank.

![Diagram of a tank](image-url)
I've drawn the tank in cross-section as a semicircle of radius 2 extending from \( y = -2 \) to \( y = 0 \).

Divide the volume of water up into circular slices. The radius of a slice is \( r = \sqrt{4 - y^2} \), so the volume of a slice is \( dV = \pi r^2 \, dy = \pi(4 - y^2) \, dy \). The weight of a slice is \( 62.4\pi(4 - y^2) \, dy \), where I'm using 62.4 pounds per cubic foot as the density of water.

To pump a slice out of the top of the tank, it must be raised a distance of \(-y\) feet. (The “-” is necessary to make \( y \) positive, since \( y \) is going from \(-2\) to \(0\).)

The work done is

\[
W = \int_{-2}^{0} 62.4\pi(-y)(4 - y^2) \, dy = 62.4\pi \int_{-2}^{0} (y^3 - 4y) \, dy = 62.4\pi \left[ \frac{1}{4}y^4 - 2y^2 \right]_{-2}^{0} = 249.6\pi \approx 784.14153 \text{ foot} - \text{pounds}. \quad \square
\]

18. Let

\[
x = \frac{\sqrt{3}}{2} t^2, \quad y = t - \frac{1}{4} t^3.
\]

Find the length of the arc of the curve from \( t = -2 \) to \( t = 2 \).

\[
\frac{dx}{dt} = \sqrt{3} \quad \text{and} \quad \frac{dy}{dt} = 1 - \frac{3}{4} t^2,
\]

so

\[
\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = \left( \sqrt{3} \right)^2 + \left( 1 - \frac{3}{4} t^2 \right)^2 = 3t^2 + \left( 1 - \frac{3}{4} t^2 \right)^2 = 3t^2 + 1 - \frac{3}{2} t^2 + \frac{9}{16} t^4 = 1 + \frac{3}{2} t^2 + \frac{9}{16} t^4 = \left( 1 + \frac{3}{4} t^2 \right)^2.
\]

Therefore,

\[
\sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} = 1 + \frac{3}{4} t^2.
\]

The length is

\[
\int_{-2}^{2} \left( 1 + \frac{3}{4} t^2 \right) \, dt = \left[ t + \frac{1}{4} t^3 \right]_{-2}^{2} = 8. \quad \square
\]

19. Find the area of the surface generated by revolving \( y = \frac{1}{3} x^3 \), \( 0 \leq x \leq 2 \), about the \( x \)-axis.

The derivative is

\[
\frac{dy}{dx} = x^2, \quad \text{so} \quad \sqrt{\left( \frac{dy}{dx} \right)^2 + 1} = \sqrt{x^4 + 1}.
\]
The curve is being revolved about the $x$-axis, so the radius of revolution is $R = y = \frac{1}{3}x^3$. The area of the surface is

$$S = \int_0^2 2\pi \left(\frac{1}{3}x^3\right) \sqrt{x^4 + 1} \, dx = \frac{2\pi}{3} \int_1^{17} u^{1/2} \cdot x^3 \left(\frac{4u}{3x^3}\right) = \pi \frac{2}{3} \left[\frac{y^{3/2}}{y^{3/2}}\right]_1^{17} = \frac{\pi}{9} \left(17^{3/2} - 1\right) \approx 24.11794.$$

20. (a) Convert $(x - 3)^2 + (y + 4)^2 = 25$ to polar and simplify.

$$(x - 3)^2 + (y + 4)^2 = 25, \quad x^2 - 6x + 9 + y^2 + 8y + 16 = 25, \quad x^2 + y^2 = 6x - 8y,$$

$$r^2 = 6r \cos \theta - 8r \sin \theta, \quad r = 6 \cos \theta - 8 \sin \theta.$$

(b) Convert $r = 4 \cos \theta - 6 \sin \theta$ to rectangular and describe the graph.

$$r = 4 \cos \theta - 6 \sin \theta, \quad r^2 = 4r \cos \theta - 6r \sin \theta, \quad x^2 + y^2 = 4x - 6y, \quad x^2 - 4x + y^2 + 6y = 0,$$

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 13, \quad (x - 2)^2 + (y + 3)^2 = 13.$$

The graph is a circle of radius $\sqrt{13}$ centered at $(2, -3)$.

21. Find the area of the region inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 3 \cos \theta$. 
Find the intersection points:

\[ 3 \cos \theta = 1 + \cos \theta, \quad 2 \cos \theta = 1, \quad \cos \theta = \frac{1}{2}, \quad \theta = \pm \frac{\pi}{3}. \]

I’ll find the area of the shaded region and double it to get the total. The shaded area is

\[
\left( \text{cardioid area from } \frac{\pi}{3} \text{ to } \pi \right) - \left( \text{circle area from } \frac{\pi}{3} \text{ to } \frac{\pi}{2} \right).
\]

The cardioid area is

\[
\int_{\pi/3}^{\pi} \frac{1}{2} (1 + \cos \theta)^2 \, d\theta = \frac{1}{2} \int_{\pi/3}^{\pi} (1 + 2 \cos \theta + (\cos \theta)^2) \, d\theta = \frac{1}{2} \int_{\pi/3}^{\pi} \left(1 + 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta)\right) \, d\theta = \\
\frac{1}{2} \left[ \theta + 2 \sin \theta + \frac{1}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) \right]_{\pi/3}^{\pi} = \frac{\pi}{2} - \frac{9}{16} \sqrt{3}.
\]

The circle area is

\[
\int_{\pi/3}^{\pi/2} \frac{1}{2} (3 \cos \theta)^2 \, d\theta = \frac{9}{4} \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) \, d\theta = \frac{9}{4} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_{\pi/3}^{\pi/2} = \frac{3\pi}{8} - \frac{9}{16} \sqrt{3}.
\]

Thus, the shaded area is

\[
\left( \frac{\pi}{2} - \frac{9}{16} \sqrt{3} \right) - \left( \frac{3\pi}{8} - \frac{9}{16} \sqrt{3} \right) = \frac{\pi}{8}.
\]

The total area is \(2 \cdot \frac{\pi}{8} = \frac{\pi}{4} \approx 0.78540\).  

_The best thing for being sad is to learn something._ - Merlyn, in T. H. White’s _The Once and Future King_